

# Polyakov formulas for GJMS operators from AdS/CFT

---

**Danilo E. Díaz**

*Humboldt-Universität zu Berlin, Institut für Physik,  
Newtonstr.15, D-12489 Berlin, Germany  
E-mail: ddiaz@physik.hu-berlin.de*

**ABSTRACT:** We argue that the AdS/CFT calculational prescription for double-trace deformations leads to a holographic derivation of the conformal anomaly, and its conformal primitive, associated to the whole family of conformally covariant powers of the Laplacian (GJMS operators) at the conformal boundary. The bulk side involves a quantum 1-loop correction to the SUGRA action and the boundary counterpart accounts for a sub-leading term in the large- $N$  limit. The sequence of GJMS conformal Laplacians shows up in the two-point function of the CFT operator dual to a bulk scalar field at certain values of its scaling dimension. The restriction to conformally flat boundary metrics reduces the bulk computation to that of volume renormalization which renders the universal type A anomaly. In this way, we directly connect two chief roles of the Q-curvature: the main term in Polyakov formulas on one hand, and its relation to the Poincare metrics of the Fefferman-Graham construction, on the other hand. We find agreement with previously conjectured patterns including a generic and simple formula for the type A anomaly coefficient that matches all reported values in the literature concerning GJMS operators, to our knowledge.

**KEYWORDS:** AdS-CFT Correspondence, Anomalies in Field and String Theories.

---

## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The physical motivation: AdS/CFT correspondence</b>	<b>5</b>
2.1 The generalized prescription for double-trace deformations	5
2.2 Bulk one-loop effective actions	6
2.3 Boundary partition function	6
2.4 The functional determinants	7
<b>3. Digression on regularizations</b>	<b>8</b>
<b>4. The general case of Poincare metrics</b>	<b>11</b>
<b>5. Conformally flat class and volume renormalization</b>	<b>12</b>
5.1 The infinitesimal variation: conformal anomaly	13
5.2 Type A holographic anomaly	13
5.3 Conformal primitive: Polyakov formulas	14
<b>6. Comparison to “experiment”</b>	<b>14</b>
6.1 Further data for the conformal Laplacian	16
<b>7. Conclusion</b>	<b>17</b>
<b>A. GJMS operators and Q-curvature</b>	<b>19</b>
<b>B. Q-curvature and volume renormalization</b>	<b>20</b>
<b>C. Rigid case in dimensional regularization</b>	<b>22</b>

---

## 1. Introduction

Conformally covariant differential operators have been the subject of a continuous interplay between physics and mathematics ever since the discovery of conformal invariance of Maxwell’s equations by Cunningham [1] and Bateman [2] in the early part of last century. To this early list belongs the Dirac operator, after Pauli’s proof of the conformal invariance of the massless Dirac equation [3], as well as the conformal wave operator, both in curved spacetime. The Riemannian variant of the later, the conformal Laplacian, is best known to mathematicians for its role in the Yamabe problem of prescribing scalar curvature on a Riemannian manifold [4].

In the early 80's a fourth-order conformal covariant was found by Paneitz [5], and independently by Eastwood and Singer [6], in relation with gauge fixing Maxwell equations respecting conformal symmetry. It was rediscovered by Riegert [7] while pursuing a different goal, namely a four-dimensional analog of Polyakov formula [8] for the conformal (or trace, or Weyl) anomaly, i.e. a non-local covariant action whose conformal variation leads to the general form of the anomaly. Graham, Jenne, Mason and Sparling [9] further showed that the conformal Laplacian and the Paneitz operator generalize to a family of conformally covariant differential operators  $P_{2k}$  of even order  $2k$ , with leading term  $\Delta^k$ , whenever the dimension  $d$  of the manifold is odd or  $d \geq 2k$ . These 'conformally covariant powers of the Laplacian' ("GJMS" operators in what follows)  $P_{2k}$  were obtained using the Fefferman-Graham ambient metric [10], a chief tool for the systematic construction of conformal invariants.<sup>1</sup>

The feature of the GJMS operators that we will treat in this paper is the conformal variation of their functional determinant<sup>2</sup> encoded in a (*generalized*) *Polyakov formula*. In other words, we focus on the conformal anomaly, and its *conformal primitive*, associated to this family of operators. These formulas can be worked out case by case in low dimensions from heat kernel coefficients whose complexity grows significantly with the dimension of the compact manifold  $\mathcal{M}$ . However, Branson [13] succeeded in finding a pattern in terms of the *Q-curvature*, in the conformally flat case, to rewrite 'more invariantly' the quotient of functional determinants of a conformally covariant operator  $A$  (or a power thereof and with suitable positive ellipticity properties) at conformally related metrics  $\hat{g} = e^{2w}g$  in any even dimension  $d$ :

$$-\log \frac{\det \hat{A}}{\det A} = c \int_{\mathcal{M}} w \left( \hat{Q}_d dv_{\hat{g}} + Q_d dv_g \right) + \int_{\mathcal{M}} \left( \hat{F} dv_{\hat{g}} - F dv_g \right) + (\text{global term}) \quad (1.1a)$$

$$= 2c \int_{\mathcal{M}} w \left( Q_d + \frac{1}{2} P_d w \right) dv_g + \int_{\mathcal{M}} \left( \hat{F} dv_{\hat{g}} - F dv_g \right) + (\text{global term}). \quad (1.1b)$$

Here  $F$  stands for a local curvature invariant and the global term is related to the null space (kernel) of the operator; both vary depending on what  $A$  is, whereas the universal part is captured by the Q-curvature term. Away from conformal flatness, the pattern is preserved in  $d = 2, 4, 6$  and conjectured to hold in all even dimension. A related conjecture by Deser and Schwimmer [14] expresses the infinitesimal variation (anomaly) as a combination of the Euler density (or *Pfaffian*), a local conformal invariant and a total derivative;<sup>3</sup> but it can be rephrased in terms of the Q-curvature instead of the Pfaffian.

It was in this context that Branson first defined the Q-curvature, in general even dimension, from the zeroth order (in derivatives) term of the GJMS operators via analytic continuation in the dimension. The *linear* transformation law under conformal rescaling  $\hat{g} = e^{2w}g$  of the Q-curvature generalizes that of the scalar curvature in two dimensions,

$$e^{d w} \hat{Q}_d = Q_d + P_d w, \quad (1.2)$$

---

<sup>1</sup>In a physical setting, see the recent work [11, 12] for an alternative route.

<sup>2</sup>In the mathematical literature, the zeta-regularized functional determinant is usually meant.

<sup>3</sup>See [15] and [16] for independent proofs. The restriction to the conformally flat class, where the Weyl tensor vanishes, was anticipated in [17].

and integrates to a multiple of the Euler characteristic on conformally flat manifolds.

Another context in which the Q-curvature plays a central role arises in the volume renormalization of conformally compact asymptotically Einstein manifolds [18], i.e. manifolds whose filling metric is the Poincare metric of the Fefferman-Graham construction. The renewed interest in the seminal work of Fefferman and Graham [10], as an outgrowth of the relation between the geometry of hyperbolic space (Lobachevsky space) and conformal geometry on the sphere at the conformal infinity, has been triggered by the AdS/CFT Correspondence in physics [19–21]. The reconstruction of a *bulk metric* associated to a given *conformal structure* at the conformal infinity [22] and the subsequent evaluation of the Einstein action with a negative cosmological term becomes the geometrical task in the limit in which the effective description of string theory is featured by the classical supergravity approximation. When the bulk metric is Einstein, the Lagrangian factorizes and the challenge consists in the regularization of the infinite (*infrared divergent*) volume. A key feature of the AdS/CFT duality is the fact that infrared divergences in the bulk are related to ultraviolet ones on the boundary theory, the so called *IR-UV connection* [23]. A further elaboration thereof leads to the mapping of the conformal anomaly of the gauge theory on the boundary, at large  $N$  (rank of the gauge group) and large 't Hooft coupling, to the failure of the renormalized bulk action to be independent of the conformal representative of the metric at the boundary. This analysis was thoroughly carried out by Henningson and Skenderis [24]. The *holographic anomaly* associated to the volume is given by the coefficient  $v^{(d)}$  of the volume expansion and its integral gives the coefficient  $\mathcal{L}$  of the log-divergent term in the volume asymptotics. The Q-curvature enters here after the observation by Graham and Zworski [25] that the “integrated anomaly”  $\mathcal{L}$  is also proportional to the integral of the Q-curvature. A further refinement by Graham and Juhl [26] results in a *holographic formula* for the Q-curvature in terms of the coefficients of the volume expansion.

These later developments involve scattering theory for Poincare metrics associated to the conformal structure as an alternative route to GJMS operators and to Q-curvature [25, 27]. They are influenced by, and at the same time generalize, results originally discussed in the physical context by Witten [21] for the “rigid” case of hyperbolic space. They are also closely related to the AdS/CFT computation of *matter* conformal anomalies which show up in the trace of the holographic energy-momentum tensor, in addition to the purely geometric terms involving curvature invariants of the boundary metric, for certain non-trivial profiles of the matter fields. Here, the powers of the Laplacian that arise in the rigid case [28–30] naturally generalize to the GJMS operators, as outlined in [29, 30]. In particular, the GJMS Laplacians show up as residues of the scattering operator  $S_M(\lambda)$  at the poles  $\lambda = d/2 + k$ ,  $k \in \mathbb{N}$ . The “rigid” version of the scattering operator  $S_M(\lambda)$  is the two-point function  $\langle O_\lambda O_\lambda \rangle$  of a CFT operator  $O_\lambda$  of conformal dimension  $\lambda$  on  $\mathbb{R}^d$  or  $\mathbb{S}^d$  as conformal boundary of the half-space model or of the ball model of the hyperbolic space  $\mathbb{H}^{d+1}$ , respectively.

In the mapping of anomalies at leading large  $N$ , the coupling constant regimes in which the bulk and boundary computations are done do not overlap; an underlying non-renormalization of the coefficient of the Euler term (and, therefore, of the Q-curvature) in the anomaly at leading large  $N$  supports the successful matching in two dimensions. The

same is true for the coefficients of the Euler and Weyl terms in four dimensions, where the free field computation on the boundary involves a combination of functional determinants of Laplacians on forms, depending on the field content of the multiplet. However, in six dimensions the coefficient of the Euler density is no longer protected and agreement between the holographic and the free CFT anomaly computation is found only for the Weyl terms [31]. Moreover, the  $AdS_9$  holographic conformal anomaly [32] is still waiting for a  $CFT_8$  description. In consequence, there seems to be little hope to get, via a holographic procedure involving the classical SUGRA action, the anomaly associated to individual differential operators (e.g. conformal Laplacian) in generic even dimension  $d$ , let alone the full determinant, and to directly connect the Q-curvature terms that naturally arise in both contexts.

In this note, we propose a heuristic “holographic” derivation of the Polyakov formulas for the GJMS operators in the conformally flat class (1.1). It is based on a remarkable prediction of AdS/CFT Correspondence, verified in the “rigid” case of hyperbolic space as bulk metric [33–37], relating corrections to the partition functions due to a relevant *double-trace deformation* of the CFT, namely a quantum 1-loop in the bulk and a next-to-leading contribution in the large- $N$  expansion at the boundary. The general situation will involve  $(X; g_+)$  as a  $d+1$  dimensional manifold with a Poincare metric and  $(M; [g])$  as its conformal infinity. Our working formula will then be the natural generalization of the formal equality that was shown to be valid in dimensional regularization in the rigid situation [37]:

$$\log \frac{\det_+[\Delta_X - \lambda(d - \lambda)]}{\det_-[\Delta_X - \lambda(d - \lambda)]} = -\log \det S_M(\lambda) . \tag{1.3}$$

The determinants of the positive Laplacian on the bulk  $X$  are evaluated using the Green’s function method, involving the resolvent at  $\lambda$  for the ‘+branch’ and its analytic continuation at  $d-\lambda$  for the ‘-branch’. The continuation in the spectral parameter  $\lambda$  to  $\lambda = d/2+k$ ,  $k \in \mathbb{N}$  as argument of the scattering operator  $S_M$  on the compact boundary  $M$  renders then the functional determinant of the GJMS conformal Laplacians. The crucial point that simplifies our present computation is that when restricted to the conformally flat class, i.e. metrics on  $M$  conformal to the standard round metric on  $S^d$ , the bulk  $X$  remains (isometric to) the hyperbolic space  $\mathbb{H}^{d+1}$  [22, 38]. In this case, the volume of the hyperbolic space factorizes and the task is then reduced to volume renormalization, with the only restriction of conformal flatness of the boundary metric. We focus on the conformal anomaly, which is then traced back to the holographic anomaly of the renormalized volume. The usual Hadamard regularization of the volume produces an anomaly and a corresponding Polyakov formula which differs from those obtained by standard zeta-regularization on the boundary, but the *universal* type A anomalous term associated to the Pfaffian or to the Q-curvature does agree.

The paper is organized as follows: we start with the physical motivation for the functional determinant identity coming from the generalized AdS/CFT prescription to treat double-trace deformations of the boundary conformal theory. We then review the rigid case determinants in the light of several possible regularization techniques. Next we state the conjectured equality between functional determinants in the general case of a filling Poincare metric with prescribed conformal infinity. Explicit computations are then presented in the case of conformally flat boundary metrics for both infinitesimal and finite

conformal variations. For comparison, we collect then all reported Polyakov formulas for GJMS operators in the literature, to our knowledge. We finally conclude by summarizing our holographic findings and hint at possible further extensions. Some background material is collected in three appendices.

## 2. The physical motivation: AdS/CFT correspondence

The celebrated Maldacena’s conjecture [19] and its calculational prescription [20, 21] entail the equality between the partition function of String/M-theory (with prescribed boundary conditions) in the product space  $AdS_{d+1} \times Y$ , for some compact space  $Y$ , and the generating functional of the dual  $CFT_d$  at the conformal boundary. One of the most remarkable successes of this duality is the mapping of the conformal anomaly at leading large  $N$  [24], as an outgrowth of the IR-UV connection [23], that relates the classical supergravity (SUGRA) action in the bulk to a quantum one-loop anomaly on the boundary.

The rank  $N$  of the gauge group measures the size of the geometry in Planck units ( $L_{AdS}/l_P = N^{1/4}$ ), implying that quantum corrections to this classical SUGRA action correspond to subleading terms in the large  $N$  limit of the CFT. At this level, there is a universal AdS/CFT result, not relying on SUSY or any other detail encoded in the compact space  $Y$ , concerning an  $O(1)$  correction to the conformal anomaly under a flow produced by a double-trace deformation. This correction was first computed in the bulk of AdS [33] and confirmed shortly after by a field theoretic computation on the dual boundary theory [34] (see also [35, 36]). Full agreement was finally shown with the help of dimensional regularization in [37], where we were able to match the anomaly as well as the renormalized values of the functional determinants involved. In the rest of this section we briefly survey these preliminary results.

### 2.1 The generalized prescription for double-trace deformations

A subtle example of the duality involves a scalar field  $\phi$  with “tachyonic” mass in the window  $-\frac{d^2}{4} \leq m^2 < -\frac{d^2}{4} + 1$ , first considered long ago by Breitenlohner and Freedman [39]. Two AdS-invariant quantizations are known to exist, since one may fix either the faster or slower falloff of the quantum fluctuations of the scalar field at infinity. The modern AdS/CFT interpretation [40] assigns the same bulk theory to two different dual CFTs in which the field  $\phi$  is dual to an operators of dimension  $\lambda_-$  and  $\lambda_+$ , respectively, and whose generating functionals are related to each other by Legendre transformation at leading large  $N$ . The conformal dimensions of the dual CFT operators, given by the two roots  $\lambda_{\pm} = \frac{d}{2} \pm \nu$  (with  $\nu = \sqrt{\frac{d^2}{4} + m^2}$ ) of the AdS/CFT relation  $m^2 = \lambda(\lambda - d)$ , are then both above the unitarity bound.

The generalized AdS/CFT prescription to incorporate boundary multi-trace operators [41] provides a dynamical picture: a boundary condition on the bulk scalar relating linearly the faster falloff part to the slower one corresponds to a double-trace deformation of the CFT Lagrangian. The two CFTs of above are then the end points of a RG flow triggered by the relevant perturbation  $f O_{\alpha}^2$  of the  $\alpha$ -CFT, where the operator  $O_{\alpha}$  has dimension  $\lambda_-$ . The  $\alpha$ -theory in the UV flows into the  $\beta$ -theory in the IR, which now has

an operator  $O_\beta$  with dimension  $\lambda_+ = d - \lambda_-$  conjugate to  $\lambda_-$ . The rest of the operators remains untouched at leading large  $N$ , which from the bulk perspective suggests that the metric and the rest of the fields involved should retain their background values, only the dual bulk scalar changes its asymptotics.<sup>4</sup>

## 2.2 Bulk one-loop effective actions

Since the only change in the bulk is in the asymptotics of the scalar field, pure AdS with zero scalar field remains a solution of the theory at the classical level for arbitrary linear boundary conditions on the scalar field. This means that for this trivial scalar field profile the effect on the partition function cannot be seen at the classical gravity level in the bulk, i.e. at leading large  $N$ . However, the contribution of the quantum fluctuations of the scalar field, given by the functional determinant of the kinetic term (inverse propagator), are certainly sensitive to the boundary conditions (reminiscent of the Casimir effect). In particular, at the endpoints of this RG flow there are two different propagators  $G_{\lambda_-}$  and  $G_{\lambda_+}$  corresponding to the two different AdS-invariant quantizations by fixing the faster or the slower falloff, respectively. The partition function including the one-loop back-reaction of the scalar field is given by

$$Z_{\text{grav}}^\pm = Z_{\text{grav}}^{\text{class}} \cdot [\det_\pm(-\square + m^2)]^{-\frac{1}{2}}, \tag{2.1}$$

where  $Z_{\text{grav}}^{\text{class}}$  refers to the saddle point approximation, i.e. the classical action. Using the Green's functions to compute the functional determinants, one realizes that no UV infinities show up in the ratio  $Z_{\text{grav}}^+/Z_{\text{grav}}^-$ , since the UV-divergences can be controlled exactly in the same way for both propagators. In addition, due to the homogeneity of AdS, the volume is factorized so that the only divergence in the ratio of one-loop corrected partition functions is the IR one given by the infinite volume of AdS.

The situation is now analog to the leading large  $N$  computation of the CFT conformal anomaly, where the classical Lagrangian density factorizes and is responsible for the  $N^2$  factor, and the geometric part is produced by the regularization of the IR-divergent volume of the bulk. In consequence, from the above correction to the classical gravitational action (relative change of the effective cosmological term) one can read off an  $O(1)$  contribution to the integrated *holographic conformal anomaly*, or equivalently to the CFT central charge, as predicted by Gubser and Mitra [33].

## 2.3 Boundary partition function

The analysis of the corresponding effect on the boundary, as done by Gubser and Klebanov [34], starts by turning on the deformation  $f O_\alpha^2$  in the  $\alpha$ -theory. Then the Hubbard-Stratonovich transformation (i.e. auxiliary field trick) can be used to linearize in  $O_\alpha$

$$\langle e^{-\frac{f}{2} \int O_\alpha^2} \rangle \sim \int \mathcal{D}\sigma e^{\frac{1}{2f} \int \sigma^2} \langle e^{\int \sigma O_\alpha} \rangle. \tag{2.2}$$

---

<sup>4</sup>The simplest realization of this behavior being the  $O(N)$  vector model in  $2 < d < 4$ , see e.g. [42, 43].

Now the large- $N$  factorization, which means that the correlators are dominated by the product of two-point functions, is explicitly used to write

$$\langle e^{\int \sigma O_\alpha} \rangle_{N \gg 1} \approx e^{\frac{1}{2} \int \int \sigma \langle O_\alpha O_\alpha \rangle \sigma} . \quad (2.3)$$

Finally, integrating back the auxiliary field produces its fluctuation determinant  $\Xi^{-1/2} = (\frac{1}{f} + \langle O_\alpha O_\alpha \rangle)^{-1/2}$ . The  $\beta$ -CFT is reached in the limit  $f \rightarrow \infty$ , so that modulo unimportant constant factors

$$Z_\beta / Z_\alpha = [\det \langle O_\alpha O_\alpha \rangle]^{-\frac{1}{2}} = [\det \langle O_\beta O_\beta \rangle]^{\frac{1}{2}} . \quad (2.4)$$

Gubser and Klebanov [34] were able to isolate the coefficient of the Euler term in the conformal anomaly of the above functional determinant on the round  $d$ -sphere. Explicit computations for several values of the dimension ( $d = 2, 4, 6, 8$ ) produced the same polynomials in  $\nu$  as those “holographically” predicted by the  $AdS_{d+1}$  computation.

In principle, the holographic recipe should work for any value of the deformation parameter  $f$ . However, away from the conformal choices ( $f = 0, \infty$ ), i.e. at an intermediate stage of the RG flow, the bulk scalar propagator at coincident points depends on the radial position and, therefore, the volume factorization breaks down. This makes the bulk one-loop computation technically more difficult and so far there are no explicit results available, only the monotonicity of the RG flow has been established [33].

## 2.4 The functional determinants

The AdS/CFT correspondence claims the equality between the partition functions. In particular, at the one-loop level in the bulk and at the corresponding subleading large- $N$  order on the boundary, this implies the equality between the functional determinants (2.1) and (2.4) involved in  $Z_{\text{grav}}^+ / Z_{\text{grav}}^- = Z_\beta / Z_\alpha$ . As supporting evidence, Hartman and Rastelli [36] gave a “kinematical” explanation. But, as usual in AdS/CFT correspondence, the prediction is a formal relation between divergent quantities which has to be properly regularized to make sense out of it. The only success so far was the correct mapping of the integrated trace anomaly.<sup>5</sup>

Finally, in [37] Dorn and the present author were able to find full agreement between the dimensionally regularized functional determinants for generic even and odd dimension  $d$ . Dimensional regularization (DR) proved to be a sensible scheme that puts bulk and boundary divergences on equal footing. We extended the mapping from that of the integrated anomaly to the renormalized partition functions as well (cf. appendix C). The anomaly can be read as the residue of the pole term. It is important to emphasize that to correctly reproduce the boundary answer, the contribution from the renormalized volume  $\mathcal{V}_{d+1}$  is not enough and there is an additional term, non-polynomial in  $\nu$ , multiplying

---

<sup>5</sup>We have been a little cavalier here since the Breitenlohner-Freedman analysis is done in Lorentzian signature. However, for computational purposes it is easier to consider the Euclidean formulation of the CFT and the volume renormalization with Riemannian signature, so that a Wick rotation should be performed. The Feynman propagator for the regular modes ( $\lambda_+$ ) in  $AdS_{d+1}$  becomes the resolvent in  $\mathbb{H}^{d+1}$ , whereas the continuation to hyperbolic space of the propagator for the irregular modes is only achieved via the continuation of the resolvent from  $\lambda_+$  to  $\lambda_-$  [44].



the integrated anomaly  $\mathcal{L}_{d+1}$  of the renormalized volume. This means that the “naive” volume factorization is not quite correct, as explained in [37], and some combination of UV-vanishing terms from the effective potential and IR-divergent ones from the volume is needed. However, due to the conformal invariance of  $\mathcal{L}_{d+1}$ , we can ignore this additional term as long as we are interested in the variation under conformal transformations as in the case of the Polyakov formulas.

### 3. Digression on regularizations

The successful mapping of the bulk and boundary computations in dimensional regularization is guaranteed by the common regularization tool. It remains unclear which regularization on the boundary corresponds to the usual cutoff (Hadamard) regularization of the volume in the bulk. If the regularization schemes of the bulk and boundary determinants are not on equal footing, only the universal terms in the anomaly (type A and B) are guaranteed to coincide. Before embarking ourselves into the study of the behavior under conformal transformations, let us first illustrate the rigid computation with different regularization schemes in the simple case of the (positive) Laplacian<sup>6</sup> in two dimensions ( $d = 2, \nu = 1$ ). The raw determinant, involving the eigenvalues  $l(l + 1)$  with multiplicity  $2l + 1$  is simply given by

$$\log \det \Delta = \text{tr} \log \Delta = \sum_{l=1}^{\infty} (2l + 1) \log [l(l + 1)], \tag{3.1}$$

where we have excluded the zero eigenvalue corresponding to constant functions.

**Bulk and boundary: dimensional regularization.** The full agreement between bulk and boundary computations using dimensional regularization  $D = 2 - \epsilon$  as shown in [37], results in a divergent term and a finite remnant as  $\epsilon \rightarrow 0$ , given by (C.5) and (C.6) respectively. As  $\nu \rightarrow 1$ , there is an additional divergence due to the zero mode but it happens on both sides of the equality and can be removed by starting the sum at  $l = 1$ . After this cancelation and within a “minimal subtraction” prescription, where we just drop the pole, we get a renormalized value for the functional determinant

$$\log \det \Delta = -4\zeta'(-1) - \frac{2}{3}\gamma + \frac{1}{6}. \tag{3.2}$$

The integrated anomaly on the two sphere can be directly read from the pole term. Here  $\mathcal{L}_3$  happens to be conformal invariant only at the integer dimension  $d = 2$ . Away from it, at  $D = 2 - \epsilon$  and under constant rescaling  $g \rightarrow \hat{g} = e^{2\alpha}g$ , it picks up a factor  $e^{(D-2)\alpha} = e^{-\epsilon\alpha}$  so that in the limit  $\epsilon \rightarrow 0$  the variation has a zero  $e^{-\epsilon\alpha} - 1$  that cancels exactly the pole and we end up with minus the residue. But the variation of the finite remnant, the renormalized value, is just minus the variation of the divergent part that

---

<sup>6</sup>An immediate check of the connection with GJMS operators comes from the direct evaluation of the eigenvalue  $\frac{\Gamma(l+\frac{d}{2}+\nu)}{\Gamma(l+\frac{d}{2}-\nu)}$  of the intertwiner (see C.3) at integer values  $\nu = k$ . These are precisely the eigenvalues of the GJMS operator  $P_{2k}$  on the standard sphere (cf. theo.2.8(f) in [13], or [45]).

is thrown away in the renormalization prescription. In all, the contribution of the local curvature invariants to the integrated conformal anomaly in the present case is just

$$\mathcal{L}_3 \cdot \left[ - \int_0^1 dx 4x \mathcal{A}_2(x) \right] = -\frac{2}{3}. \tag{3.3}$$

Alternatively, one can use the fact that in the renormalized result (finite remnant) at  $d = 2$  only the term proportional to the renormalized volume  $\mathcal{V}_3$  is anomalous. The integrated anomaly corresponding to the renormalized volume is known to be just  $\mathcal{L}_3$ . Keeping track on the coefficient, we find again the same result as above.

However, there is still a missing contribution (a global term) from the exclusion of the zero mode, as  $g \rightarrow \hat{g} = e^{2\alpha}g$  the eigenvalues are also rescaled  $\lambda_l \rightarrow \hat{\lambda}_l = e^{-2\alpha}\lambda$  so that we must consider:

$$-2\alpha \sum_{l=1}^{\infty} \text{deg}(D, l). \tag{3.4}$$

In dimensional regularization, the sum over degeneracies starting with  $l = 0$  vanishes, so that the above renders an additional contribution  $2\alpha \text{deg}(D, 0) = 2\alpha$ . In all,

$$\log \frac{\det \hat{\Delta}}{\det \Delta} = -\frac{2}{3}\alpha + 2\alpha = \frac{4}{3}\alpha. \tag{3.5}$$

This totally agrees with the Polyakov formula evaluated for a constant rescaling  $w = \alpha$ . The first contribution  $-\frac{2}{3}\alpha$  comes from the Q-curvature term  $2c \int_{\mathcal{M}} w Q_n dv_g$ , which is precisely the *conformal index* of Branson and Ørsted [46], and the remaining  $2\alpha$  corresponds to the global term associated to the zero mode. From the rigid computation on the sphere we can therefore read the coefficient of the Q-curvature term (type A anomaly) as well as the global term. We get no information, however, on the local invariant term  $F$  in the Polyakov formula because it is scale invariant and, therefore, under rigid rescaling doesn't show up.

The corresponding extension of this computation to higher dimensions and to the whole family of GJMS operators is straightforward in dimensional regularization.

**Boundary: zeta-regularization.** The zeta regularization produces directly a renormalized determinant

$$\log \det \Delta = -\zeta'_{\Delta}(0), \tag{3.6}$$

in terms of the zeta function on the two-sphere

$$\zeta_{\Delta}(s) = \sum_{\lambda>0} \lambda^{-s} = \sum_{l=1}^{\infty} \frac{2l+1}{[l(l+1)]^s}. \tag{3.7}$$

The above representation is valid for  $Re(s) > 1$ , the analytic continuation to  $s = 0$  can be accomplished by rewriting in terms of better studied zeta functions, such as Riemann and Hurwitz zeta functions. The result in the present case can be shown to be [47]

$$\log \det \Delta = -4\zeta'(-1) + \frac{1}{2}. \tag{3.8}$$

The integrated anomaly can be read in this case from the variation under rigid rescaling  $\log \det \widehat{A} - \log \det A = -2\alpha \zeta_A(0)$ . The input we need is the zeta function  $\zeta_\Delta(0) = -\frac{2}{3}$  to finally get

$$\log \frac{\det \widehat{\Delta}}{\det \Delta} = \frac{4}{3} \alpha . \tag{3.9}$$

Extensions of this computation to higher dimensions can be attacked with methods similar to those of [48]. The calculations are rather lengthy and explicit results, to our knowledge, do not cover the Paneitz operator or any other of the higher GJMS operators.

**Boundary: large-eigenvalue cutoff.** Finally, we want to present yet another regularization tool which is simply a cutoff  $l_c$  in the sum over eigenvalues. It is physically appealing due to its role regarding the holographic bound in AdS/CFT [23]: the IR-UV connection forces the number of cells in the coarse-grained sphere to be  $\epsilon^{-3}$ ; however, if one instead truncates at  $l_c$  the spherical modes, then the number of modes plays now the same role as the number of cells did before. The number of modes is given by the counting function (whose asymptotics is in general given by Weyl’s asymptotic formula) that grows as  $l_c^3$  in this case. In all, the identification  $l_c \cdot \epsilon \sim 1$  between the UV-cutoff  $l_c$  and the IR-cutoff  $\epsilon$  is enforced by the requirement of (roughly) one degree of freedom per unit Planck area.

To compute  $\log \det \Delta$  we have to consider the finite sum

$$\sum_{l=1}^{l_c} (2l+1) \log [l(l+1)] = 4 \sum_{l=1}^{l_c} l \log l + (2l_c+1) \log(l_c+1) . \tag{3.10}$$

The large  $l_c$  asymptotics of the remaining sum is determined by the Glaisher-Kinkelin constant  $A$  given by

$$1^1 2^2 3^3 \dots n^n = n^{n^2/2+n/2+1/12} e^{-n^2/2} (A + o(1)) , \tag{3.11}$$

as  $n \rightarrow \infty$ , where  $\log A = \frac{1}{12} - \zeta'(-1)$ . The total contribution for the regularized determinant is then

$$-4\zeta'(-1) + \frac{7}{3} + \left( 2l_c^2 + 4l_c + \frac{4}{3} \right) \log l_c - \frac{1}{4} l_c^2 . \tag{3.12}$$

We want to read now the integrated anomaly from the log-term, but there seems to be new divergent terms which have no analog in the volume regularization nor in heat kernel asymptotics. For example, using a “proper-time” cutoff  $\delta \rightarrow 0$ , the heat kernel produces

$$\begin{aligned} \log \det \Delta &= - \int_{\delta} \frac{dt}{t} \left\{ \frac{1}{4\pi t} \int_{S^2} \text{dvol} \left[ 1 + \frac{R}{6} t \right] - \dim \ker \Delta \right\} + \text{finite} \\ &= -\frac{1}{2\delta} + \frac{2}{3} \log \frac{1}{\delta} + \text{finite} , \end{aligned} \tag{3.13}$$

and IR-UV connection relates the cutoffs as  $\epsilon \sim \sqrt{\delta}$  in rough agreement with the volume asymptotics. Although a physical interpretation of these extra log-divergencies is still obscure to us, their structure is simple enough. They are in fact proportional to the counting function with the order of the operator as proportionality factor,  $2 \cdot \sum_{l=1}^{l_c} (2l+1) = 2 \cdot (l_c^2 + l_c)$ . We can subtract them so that the two residual divergences left are the expected

ones in view of the identification  $l_c \cdot \sqrt{\delta} \sim 1$ . In the present case, the integrated anomaly  $\frac{4}{3}$  shows up as coefficient of  $\log l_c$  and the renormalized determinant, given by

$$-4\zeta'(-1) + \frac{7}{3}, \tag{3.14}$$

correctly reproduces the ‘most transcendental’ part  $-4\zeta'(-1)$  which is common to the two previous regularization alternatives.

The above procedure can be (straightforwardly but tediously) adapted to the higher-dimensional spheres and to the whole family of GJMS. The corresponding asymptotic estimates are determined by the Glaisher-Kinkelin-Bendersky constants [49](see also [50]) in this case.

**Bulk: Hadamard regularization of the volume.** The cutoff regularization of the volume [24, 18] produces a null renormalized volume when evaluated for the standard metric of  $\mathbb{H}_3$ , so that the volume asymptotics is given by

$$\text{Vol}(\{r > \epsilon\}) = \frac{\pi}{2\epsilon^2} - 2\pi \log \frac{1}{\epsilon} + o(1). \tag{3.15}$$

When multiplied by the effective potential, the Hadamard-regularized volume will correctly reproduce the anomaly but not the finite remnant, not even the  $-4\zeta'(-1)$  piece. A compensation between divergences in the volume and vanishing terms in the effective potential, that in dimensional regularization conspired to produce the correct boundary result, is still missing here.

#### 4. The general case of Poincare metrics

After this preamble, let us now turn to the main theme. To relax the rigidity, we must first consider a generalization [21] of the AdS/CFT Correspondence in terms of a  $d + 1$ -dimensional (asymptotically) Einstein manifold  $X$  with negative cosmological constant, that has a compactification consisting of a manifold with boundary  $\bar{X}$  whose boundary points are  $M$  and whose interior points are  $X$  with a metric  $g_+$  on  $X$  that has a double pole near the boundary so that it defines a conformal structure  $[g]$  on  $M$ , i.e. the bulk metric is that of a conformally compact (asymptotically) Einstein manifold. This defines  $(X; g_+)$  as a  $d+1$  dimensional manifold with a Poincare metric and  $(M; [g])$  as its conformal infinity. Such  $g_+$  is also asymptotically hyperbolic.<sup>7</sup>

We are then naturally led to the following guess for the functional determinants involved in the one-loop bulk correction and the corresponding subleading large-N term on the boundary:

$$\log \frac{\det_+[\Delta_X - \lambda(d - \lambda)]}{\det_-[\Delta_X - \lambda(d - \lambda)]} = -\log \det S_M(\lambda). \tag{4.1}$$

Alternatively, the evaluation of the bulk determinant using the Green’s function method by taking the derivative with respect to the spectral parameter leads to the following relation

---

<sup>7</sup>In the physics context, this generalization is usually described by the somehow less rigorous notion of asymptotically (Euclidean) anti-de Sitter metrics.

in term of the resolvent  $R_X(\lambda)$  and its analytic continuation  $R_X(d - \lambda)$

$$(d - 2\lambda) \text{tr} [R_X(\lambda) - R_X(d - \lambda)] = \text{tr} \left[ S_M^{-1}(\lambda) \frac{d}{d\lambda} S_M(\lambda) \right]. \quad (4.2)$$

An analog relation has been shown to be valid by Guillarmou ([51], theo. 1.2) for certain generalized variants of the trace, but for the case in which  $d$  happens to be odd and the functional determinants are conformal invariants.

Unfortunately, the above functional determinants are too difficult to compute in general. Only in very symmetric situations they can be explicitly computed. In addition, the determinant of the scattering operator as than of an elliptic *pseudo-differential operator* is a largely unexplored object. Yet, Polyakov formulas for the ratio of functional determinants at conformally related metrics capture valuable information; they only fail to account for conformally invariant terms. To make some progress, we will then be rather interested in the variation under conformal rescaling of the boundary metric for even  $d$  and consider the continuation in the spectral parameter ( $\nu \rightarrow k$ ,  $k = 1, 2, \dots, d/2$ ) to make contact with the GJMS operators  $P_{2k}$ , which are better known, and with their corresponding Polyakov formulas. Conformal flatness of the boundary metric is not assumed in the above guess, so that under conformal rescaling both type A and type B anomalies will be present. However, in this paper we will restrict to the conformally flat situation where the bulk computation will be reduced to that of the volume renormalization as we will next show.

## 5. Conformally flat class and volume renormalization

To read the (infinitesimal) anomaly, one has to be able to compute the variation under a Weyl rescaling of the boundary metric. We let the boundary metric to be conformally related to the standard one on the sphere, so that the bulk geometry is still (isometric to) the hyperbolic space [22, 38]. In this case, the resolvent is explicitly known in terms of the hypergeometric function and one easily gets

$$[R_X(\lambda) - R_X(d - \lambda)](x, x) = 2 \mathcal{A}_d(\nu), \quad (5.1)$$

with  $\mathcal{A}_{(d)}(\nu)$ , essentially the Plancherel measure on hyperbolic space at imaginary argument, as in (C.2). There is no dependence on the position due to the homogeneity of  $\mathbb{H}_{d+1}$ , and therefore the volume factorizes when taking the trace. Integrating back in  $\nu$ , we get for the “bare” determinant

$$\left[ \int_0^k d\nu \, 2\nu \mathcal{A}_d(\nu) \right] \cdot \int_{\mathbb{H}^{d+1}} d\text{vol}_{g_+} = -\frac{1}{2} \log \det P_{2k}. \quad (5.2)$$

A renormalized version (in DR) leaves a finite remnant of the IR-divergent bulk volume, i.e. the renormalized volume  $\mathcal{V}_{d+1}$ , and additional conformally invariant terms with a non-polynomial dependence in  $k$  which play no role in the analysis of the conformal variation. The generalized Polyakov formula relating the functional determinants of the GJMS

conformal Laplacians at conformally related metrics  $\widehat{g} = e^{2w}g$  in even dimension  $d$ , up to the global term, is proportional to the conformal variation of the renormalized volume

$$-\frac{1}{2} \log \frac{\det \widehat{P}_{2k}}{\det P_{2k}} = \left[ \int_0^k d\nu \, 2\nu \mathcal{A}_d(\nu) \right] \cdot (\widehat{\mathcal{V}}_{d+1} - \mathcal{V}_{d+1}). \quad (5.3)$$

### 5.1 The infinitesimal variation: conformal anomaly

We have now to deal with the conformal variation of the renormalized volume  $\mathcal{V}_{d+1}$  as a functional of the boundary metric representative  $g$ . Up to total derivative terms, we can rely on the well known results obtained via a radial cutoff. The infinitesimal variation of the Hadamard-renormalized volume (see e.g. [18]) is given by the coefficient  $v^{(d)}$  of the volume expansion

$$\frac{d}{d\varepsilon} \mathcal{V}[e^{2\varepsilon w}g] |_{\varepsilon=0} = \int_{\mathcal{M}} w v^{(d)} dv_g, \quad (5.4)$$

i.e. a trace anomaly

$$\frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \mathcal{V}[g] = v^{(d)}. \quad (5.5)$$

To get the Q-curvature term we make then use of the holographic formula [26]

$$2c_{d/2} Q = v^{(d)} + \dots, \quad (5.6)$$

where  $c_k = (-1)^k [2^{2k}(k)!(k-1)!]^{-1}$  and the ellipsis stands for derivative terms involving lower coefficients  $v^{(k)}$ . The universal Type A anomaly of  $-\frac{1}{2} \log \det P_{2k}$ , encoded in the Q-curvature term or in the Euler term, is finally given by

$$2c_{d/2} \left[ \int_0^k d\nu \, 2\nu \mathcal{A}_d(\nu) \right] \cdot Q_d. \quad (5.7)$$

### 5.2 Type A holographic anomaly

In an independent development, the authors of [52] were able to work out the type A holographic anomaly, coefficient of the Euler term, coming from a generic gravitational action which admits AdS as solution (see also [53] for an alternative derivation). The input needed is the Lagrangian density evaluated for the AdS metric, i.e. it suffices to examine the rigid situation (in Euclidean signature) for hyperbolic space. We can now use this general result combined with the one-loop computation [37] for the rigid case, to get

$$\left[ \int_0^k d\nu \, 2\nu \mathcal{A}_d(\nu) \right] \cdot \frac{E_d}{2^d (d/2)!}. \quad (5.8)$$

Now, keeping track on the normalizations we can translate  $E_d = (-2)^{d/2} (d/2)! \text{Pff}$  to the Pfaffian, according to the conventions of [26] and further use the relation between the Pfaffian and the  $v^{(d)}$  coefficient in the conformally flat case  $v^{(d)} = \frac{(-2)^{d/2}}{(d/2)!} \text{Pff}$ . We find then full agreement with the previous result (5.7) obtained using the volume renormalization.

### 5.3 Conformal primitive: Polyakov formulas

To obtain Polyakov formulas for the quotient of the functional determinant at conformally related metrics, we have to find the conformal primitive of the infinitesimal anomaly. This we can readily do in two ways. We can apply a result by Branson for the conformal primitive of the Q-curvature term, which gives the universal part  $\int_{\mathcal{M}} w(Q_d + \frac{1}{2} P_d w) dv_g$  in Polyakov formulas.

Alternatively, via the connection between the renormalized volume and scattering theory [25] Chang, Qing and Yang [54] have found an explicit expression for  $\widehat{\mathcal{V}}_{d+1} - \mathcal{V}_{d+1}$  as conformal primitive of  $v^{(d)}$ . It contains the universal part of above and additional local curvature invariant terms to correctly reproduce the holographic formula relating  $Q_d$  and  $v^{(d)}$ .

We get then, up to the global term, for the finite conformal variation of the functional determinant

$$-\log \frac{\det \widehat{P}_{2k}}{\det P_{2k}} = c_{(d,k)} \int_{\mathcal{M}} w \left( \widehat{Q}_d dv_{\widehat{g}} + Q_d dv_g \right) + \int_{\mathcal{M}} \left( \widehat{F} dv_{\widehat{g}} - F dv_g \right) \quad (5.9a)$$

$$= 2 c_{(d,k)} \int_{\mathcal{M}} w \left( Q_d + \frac{1}{2} P_d w \right) dv_g + \int_{\mathcal{M}} \left( \widehat{F} dv_{\widehat{g}} - F dv_g \right), \quad (5.9b)$$

where

$$c_{(d,k)} = 2 c_{d/2} \left[ \int_0^k d\nu 2\nu \mathcal{A}_d(\nu) \right] = \frac{(-1)^{d/2}}{(4\pi)^{d/2} (d-1)! (d/2)!} \int_0^k d\nu (\nu)^{\frac{d}{2}} (-\nu)^{\frac{d}{2}}, \quad (5.10)$$

with  $\mathcal{A}_d(\nu)$  given as in (C.2), and the curvature invariants in the  $F$ -term, which are regularization-scheme dependent, enter with different coefficient as in the conventional zeta-regularized determinants. There is clearly a mismatch between zeta-regularization on the boundary and Hadamard regularization in the bulk, as we will see, and only the universal Q-curvature term is correctly reproduced. Any scheme that renders the Einstein-Hilbert action finite (see e.g. [30, 55]) is associated to a renormalized volume. A finite (renormalized) bulk action induces an effective action for the conformal mode on the conformal boundary which gives essentially the Polyakov formula; this has been explicitly shown in [56] and [57] where the Liouville and Riegert actions, respectively, have been obtained. The advantage of the result by Chang, Qing and Yang [54], besides being simpler and compact, is that the local curvature invariants can be explicitly derived and not only their finite variation under conformal rescaling.

## 6. Comparison to “experiment”

Here we collect all explicit results we are aware of for the GJMS Laplacians, all of them computed via standard zeta-regularization. Most of the known Polyakov formulas are due to Branson and collaborators (see e.g. [13] and references therein), whose main motivation was related to sharp inequalities and extremal problems for the functional determinant. The particular values of the coefficients and the local invariants that enter in the formulas are extracted from the relevant heat invariants, which are rewritten in the basis of the Q-curvature, the Weyl invariants plus a total derivative.

**Laplacian, d=2.** This case is the best known in physics, the original<sup>8</sup> Polyakov formula [8] for the effective action (conformal primitive) of the two-dimensional trace anomaly:

$$\begin{aligned} -\log \frac{\det \widehat{\Delta}}{\det \Delta} &= \frac{1}{24\pi} \int_{\mathcal{M}} w \left( \widehat{R} dv_{\widehat{g}} + R dv_g \right) - \log \frac{\text{vol}(\widehat{g})}{\text{vol}(g)} \\ &= \frac{1}{12\pi} \int_{\mathcal{M}} w \left( R + \frac{1}{2} \Delta w \right) dv_g - \log \frac{\text{vol}(\widehat{g})}{\text{vol}(g)}, \end{aligned} \quad (6.1)$$

In two dimensions the Q-curvature is given by the Schouten scalar which is half the scalar curvature ( $Q_2 = J = R/2$ ), and the conformal Laplacian or Yamabe operator is simply the Laplacian ( $P_2 = Y = \Delta$ ).

**Yamabe, d=4.** The result for the Yamabe operator was first obtained by Branson and Ørsted [58], although the general structure in four dimensions was anticipated by Riegert [7]. The input needed is the heat kernel coefficient  $a_4(Y)$  (see e.g. [59]), then restrict to the conformally flat class to read the coefficient of the Q-curvature as well as the local curvature invariants entering in the Polyakov formula to finally write:

$$\begin{aligned} -\frac{1}{2}(4\pi)^2 \log \frac{\det \widehat{Y}}{\det Y} &= -\frac{1}{180} \int_{\mathcal{M}} w \left( \widehat{Q} dv_{\widehat{g}} + Q dv_g \right) - \frac{1}{90} \int_{\mathcal{M}} \left( \widehat{J}^2 dv_{\widehat{g}} - J^2 dv_g \right) \\ &= -\frac{1}{90} \int_{\mathcal{M}} w \left( Q + \frac{1}{2} P w \right) dv_g - \frac{1}{90} \int_{\mathcal{M}} \left( \widehat{J}^2 dv_{\widehat{g}} - J^2 dv_g \right), \end{aligned} \quad (6.2)$$

where  $Q$  is the four-dimensional Q-curvature,  $P$  is the Paneitz operator and  $J = \frac{R}{2(d-1)}$  is the Schouten scalar.

**Paneitz, d=4.** For the Paneitz operator in four dimensions, the corresponding Polyakov formula was obtained by Branson [60] and the heat invariant input needed was computed from Gilkey's work [61],

$$\begin{aligned} -\log \frac{\det \widehat{P}}{\det P} &= \frac{1}{720\pi^2} \left[ 14 \int_{\mathcal{M}} w \left( \widehat{Q} dv_{\widehat{g}} + Q dv_g \right) - 32 \int_{\mathcal{M}} \left( \widehat{J}^2 dv_{\widehat{g}} - J^2 dv_g \right) \right] - \log \frac{\text{vol}(\widehat{g})}{\text{vol}(g)} \\ &= \frac{1}{720\pi^2} \left[ 28 \int_{\mathcal{M}} w \left( Q + \frac{1}{2} P w \right) dv_g - 32 \int_{\mathcal{M}} \left( \widehat{J}^2 dv_{\widehat{g}} - J^2 dv_g \right) \right] - \log \frac{\text{vol}(\widehat{g})}{\text{vol}(g)}. \end{aligned} \quad (6.3)$$

**Yamabe, d=6.** The six-dimensional case for the Yamabe operator was worked out by Branson [13]. The starting point is the heat kernel coefficient  $a_6$  computed by Gilkey [59], restricted to the conformally flat case.

$$\begin{aligned} -\frac{3 \cdot 7! (4\pi)^3}{2} \log \frac{\det \widehat{Y}}{\det Y} &= 5 \int_{\mathcal{M}} w \left( \widehat{Q}_6 dv_{\widehat{g}} + Q_6 dv_g \right) + 13 \int_{\mathcal{M}} \left( |\widehat{\nabla} \widehat{J}|^2 dv_{\widehat{g}} - |\nabla J|^2 dv_g \right) \\ &\quad + 34 \int_{\mathcal{M}} \left( \widehat{J}^3 dv_{\widehat{g}} - J^3 dv_g \right) - 32 \int_{\mathcal{M}} \left( \widehat{J} |\widehat{V}|^2 dv_{\widehat{g}} - J |V|^2 dv_g \right). \end{aligned} \quad (6.4)$$

Here the local invariant involves now the Schouten tensor  $V = \frac{\text{Ric} - Jg}{d-2}$  as well.

---

<sup>8</sup>These formulas ought to be called Polyakov formulas in ‘‘Liouville’s form’’. The celebrated non-local covariant form in terms of the Green’s function of  $P_d$  is obtained after eliminating the conformal factor  $w$  by solving the conformal relation  $\sqrt{g} P_d w = \sqrt{g} \widehat{Q}_d - \sqrt{g} Q_d$ .



	$d = 4$	$d = 6$	$d = 8$	$d = 10$
$\zeta_Y(0)$	$-\frac{1}{90}$	$\frac{1}{756}$	$-\frac{23}{113400}$	$\frac{263}{7484400}$

**Table 1:**  $\zeta_Y(0)$  on  $\mathbb{S}^d$

**Yamabe,  $d=8$ .** The eight-dimensional case for the Yamabe operator was worked out by Branson and Peterson [62], where the necessary input was computed from Avramidi’s result [63] for the relevant heat invariant. At this stage, the computation becomes almost prohibiting and it was only done with computer aided symbolic manipulations<sup>9</sup>

$$-\frac{(4\pi)^4}{2} \log \frac{\det \widehat{Y}}{\det Y} = -\frac{23}{1360800} \int_{\mathcal{M}} w(\widehat{Q}_8 dv_{\widehat{g}} + Q_8 dv_g) + \dots \quad (6.5)$$

### 6.1 Further data for the conformal Laplacian

As far as we are interested in the type A anomaly coefficient, we can make a longer list based on explicit computations on the spheres via zeta regularization. Once we know the zeta-function of the operator  $\zeta_A(0)$  on the sphere and the dimension  $q(A)$  of its kernel, we can work out the coefficient of the Euler term or, equivalently, the coefficient of the Q-curvature. The main relation is given by the *conformal index theorem* [46] restricted to the conformally flat class:

$$\zeta_A(0) + q(A) = \frac{c}{l} \int_M Q_d dv_g, \quad (6.6)$$

where  $2l$  is the order of the differential operator  $A$ . There is a vast literature computing the zeta function for the (conformal) Laplacian on the round sphere. They generalize early results by Weisberger [47] and are rather lengthy calculations. Remarkably, there is a compact recipe obtained by Cappelli and D’Appollonio [64] for  $d \geq 4$

$$\zeta_Y(0) = -\frac{1}{(d-1)!} \int_0^{B(1+B)} dt \prod_{i=0}^{d/2-2} [t - i(i+1)], \quad (6.7)$$

where, after performing the integral, one has to substitute the powers  $B^n$  by the Bernoulli number  $B_n$ . This produces the results in table 1.

In this case,  $q(Y)$  is 1 in two dimensions and zero for all other even dimensions

$$\zeta_Y(0) + q(Y) = c_Y(d) \Gamma(d) Vol(S^d) = c_Y(d) \Gamma(d) \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}, \quad (6.8)$$

so that we can check the coefficient of the Q-curvature  $c_Y(d) := c_{(d,1)}$  for the Polyakov formula for the conformal Laplacian (5.10) in any even dimension.

We find agreement with all these values. Our holographic result for the conformal Laplacian ( $k = 1$ ) can be easily compared to the above formula, the integral of the Q-curvature on  $\mathcal{S}^d$  is simply the volume of the sphere times the constant value  $\Gamma(d)$  of the

---

<sup>9</sup>I am indebted to L. J. Peterson for providing the coefficient of the Q-curvature term and valuable explanations.

$(d + 1)! \cdot (4\pi)^{d/2} \cdot c_{(d,k)}$					
	$k = 1$ , Yamabe	$k = 2$ , Paneitz	$k = 3$	$k = 4$	$k = 5$
$d = 2$	2 ✓	-	-	-	-
$d = 4$	-4/3 ✓	112/3 ✓	-	-	-
$d = 6$	10/3 ✓	-64/3	738	-	-
$d = 8$	-184/15 ✓	832/15	-1944/15	253184/15	-
$d = 10$	526/9 ✓	-1984/9	1026	-75776/9	4016750/9

**Table 2:** Q-curvature coefficient  $c_{(d,k)}$  in Polyakov formulas for GJMS operators

Q-curvature on the round sphere. We get then for the conformal index an even simpler formula

$$\zeta_Y(0) + q[Y] = \frac{2(-1)^{d/2}}{d!} \int_0^1 d\nu (\nu)_{\frac{d}{2}} (-\nu)_{\frac{d}{2}} = \frac{2(-1)^{d/2}}{d!} \int_0^1 d\nu \prod_{i=0}^{d/2-1} [i^2 - \nu^2]. \quad (6.9)$$

## 7. Conclusion

The main thrust of this paper has been towards a holographic derivation of Polyakov formulas for GJMS operators. On one hand, this constitutes an important test of the AdS/CFT correspondence in the general case of a Poincare metric in the bulk. The holographic description of double-trace deformations of the boundary CFT and the conjectured equality between partition function at the one-loop quantum level in the bulk and subleading large-N order on the boundary lead to a remarkable identity between functional determinants. We have been able to make progress in the case of conformally flat boundary metrics, reducing the bulk computation to that of volume renormalization. This makes, on the other hand, a direct connection between the Q-curvature that appears in the volume renormalization of Poincare metrics and the universal Q-curvature term in the Polyakov formulas for conformally covariant operators. We get a generic formula (5.10) for the type A conformal anomaly associated to the whole family of GJMS operators, in agreement with some previously known results that we summarize in table 2.

Graham [18] already noticed that the invariance properties of the renormalized volume  $\mathcal{V}$  are reminiscent of those for the functional determinant of the conformal Laplacian, which is conformally invariant in odd dimensions but which has an anomaly in even dimensions, and that the properties of the invariant  $\mathcal{L}$  are, on the other hand, similar to those for the constant term in the expansion of the integrated heat kernel for the conformal Laplacian, which vanishes in odd dimensions but in even dimensions is a conformal invariant obtained by integrating a local expression in curvature. In this note, we have gone further and shown that the above similarities can be promoted to equalities, up to regularization-scheme dependent terms. The Polyakov formulas obtained via volume renormalization correctly reproduce the universal Q-curvature term. However, the coefficients of the additional local curvature invariants are different from those obtained via  $\zeta$ -regularization on the compact

	$k = 1$ , Yamabe	$k = 2$ , Paneitz	$k = 3$	$k = 4$	$k = 5$
$d = 2$	$-\frac{2}{3}$ ✓	-	-	-	-
$d = 4$	$-\frac{1}{90}$ ✓	$-\frac{38}{45}$ ✓	-	-	-
$d = 6$	$\frac{1}{756}$ ✓	$-\frac{4}{945}$	$-\frac{379}{420}$	-	-
$d = 8$	$-\frac{23}{113400}$ ✓	$\frac{13}{28350}$	$-\frac{1}{1400}$	$\frac{562603}{604800}$	-
$d = 10$	$\frac{263}{7484400}$ ✓	$-\frac{31}{467775}$	$\frac{19}{92400}$	$-\frac{592}{467775}$	$-\frac{283309}{299376}$

**Table 3:**  $\zeta_{P_{2k}}(0)$  on  $\mathbb{S}^d$

boundary. It thus remains a challenge to find a bulk regularization that corresponds to the  $\zeta$ -regularization on the boundary.

Notwithstanding, we can unambiguously write down a compact formula for the zeta function of the GJMS operators on the round sphere

$$\zeta_{P_{2k}}(0) + \delta_{d,2k} = \frac{2}{k} \frac{(-1)^{d/2}}{d!} \int_0^k d\nu (\nu)_{\frac{d}{2}} (-\nu)_{\frac{d}{2}}, \tag{7.1}$$

which correctly reproduces all values reported in the literature and predicts new ones (table 3). It would be desirable to have a confirmation of these results; one possible way goes via the relation of the conformal anomaly with the Wodzicki residue (see e.g. [65]) and the computation of the later using symbol calculus.

We have favored the Q-curvature, rather than the Euler density, to describe the type A anomaly. This is mainly due to its simpler transformation law under conformal rescaling and its simpler conformal primitive. In conformal geometry, the Q-curvature certainly plays a central role and has been intensively studied in recent years. On the physical side, it has been less explored; however, let us mention that some purely QFT considerations regarding the irreversibility of the RG flow, unitarity and positivity of the induced action for the conformal factor and *a-theorem* have led Anselmi [66, 67] to introduce a “pondered Euler density” in the study of conformal anomalies. This “pondered Euler density” has a linear transformation law under conformal recaling, therefore it is nothing but the Q-curvature modulo Weyl invariant terms. Moreover, the explicit expression for  $d = 6$  in [67] coincides with Branson’s  $Q_6$  in six dimensions [68].

There are still several interesting issues to be explored. Going beyond conformal flatness will switch on the Weyl-terms whose number grows with the dimension. Here, the issue of uniqueness of the filling Poincare metric, together with the topology of the conformal boundary, will surely play an important role.<sup>10</sup> Even the rigid case of hyperbolic space can be extended to quotients by symmetry groups, as is the case of Kleinian groups, where connections with number theory via Selberg zeta functions naturally arise (see e.g. [71, 72] and [73], section 2.9, for a related discussion). In another direction, the relation between Polyakov formulas, extremal of functional determinants and sharp inequalities [13] may well admit a holographic interpretation in terms of the bulk geometry.

---

<sup>10</sup>In a celebrated example due to Hawking and Page [69], where the AdS Schwarzschild black hole and thermal AdS share the same conformal infinity, the bulk one-loop effect corresponding to the double-trace deformation has already been explored in [70].

## Acknowledgments

My deep gratitude to Harald Dorn for encouragement and valuable advice in preparing this manuscript. I benefited from common seminars with the group of Differential Geometry and Global Analysis at HU-Berlin given by J. Erdmenger and A. Juhl. I am also indebted to H. Baum, G. Jorjadze, L.J. Peterson and H.S. Yang for useful discussions and references. Finally, I thank the Floro Pérez Alumni Forum for the encouragement.

## A. GJMS operators and Q-curvature

To give a glimpse of these constructions in conformal geometry, let us go back to the Poincare patch and examine the analytic continuation to  $\lambda > d/2$  of the kernel

$$\frac{1}{|\vec{x}' - \vec{x}|^{2\lambda}}. \tag{A.1}$$

It will have single poles at  $\lambda = d/2 + k, k \in \mathbb{N}$ , since in the neighborhood of these values (see e.g. [74])

$$\lim_{\lambda \rightarrow d/2+k} \frac{\lambda - d/2 - k}{|\vec{x}|^{2\lambda}} = -c_k \Delta^k \delta^{(d)}(\vec{x}) \tag{A.2}$$

where

$$c_k = \frac{1}{2^{2k} k! (k-1)!}. \tag{A.3}$$

Therefore for these “resonant values” the action of the kernel reduces to that of the  $k$ -th power of the Laplacian  $\Delta^k$ , a conformal invariant (covariant) differential operator.<sup>11</sup>

The generalization of this observation [25] for a filling Poincare metric associated to a given conformal structure involves  $P_{2k}$ , the conformally invariant operators of GJMS [9].

**GJMS operators.** The GJMS operators  $P_{2k}$  built using the Fefferman-Graham ambient construction have, among others, the following properties in a  $d$ -dim Riemannian manifold  $(M, g)$

- On flat  $\mathbb{R}^d$ ,  $P_{2k} = \Delta^k$
- $P_{2k} \exists k \in \mathbb{N}$  and  $k - d/2 \notin \mathbb{Z}^+$
- $P_{2k} = \Delta^k + (\text{lower order terms})$
- $P_{2k}$  is self-adjoint
- for  $f \in C^\infty(M)$ , under a conformal change of metric  $\hat{g} = e^{2\sigma} g$ ,  $\sigma \in C^\infty(M)$ , conformal covariance:  $\hat{P}_{2k} f = e^{-\frac{d+2k}{2}\sigma} P_{2k}(e^{\frac{d-2k}{2}\sigma} f)$
- $P_{2k}$  has a polynomial expansion in  $\nabla$  and the Riemann tensor (actually the Ricci tensor) in which all coefficients are rational in the dimension  $d$
- $P_{2k}$  has the form  $\nabla \cdot (S_k \nabla) + \frac{d-2k}{2} Q_k^d$ , where  $S_k = \Delta^{k-1} + (\text{lower order terms})$  and  $Q_k^d$  is a local scalar invariant.

---

<sup>11</sup>There is a factor  $(-1)^k$  hanging around, just because in the mathematical literature the *positive Laplacian* is preferred.

**Q-curvature.** The  $Q$  – curvature generalizes in many ways the 2-dim scalar curvature  $R$ . Its original derivation tries to mimic the derivation of the *prescribed Gaussian curvature equation* (PGC) in 2-dim starting from the *Yamabe equation* in higher dimension and analytically continuing to  $d = 2$ .

Start with the conformal transformation of the scalar curvature at  $d \geq 3$

$$e^{2\sigma} \widehat{R} = R + 2(d-1)\Delta\sigma - (d-1)(d-2)\nabla\sigma \cdot \nabla\sigma \tag{A.4}$$

and absorb the quadratic term

$$\Delta\sigma - (d/2 - 1)\nabla\sigma \cdot \nabla\sigma = \frac{2}{d-2} e^{-(d/2-1)\sigma} \Delta e^{(d/2-1)\sigma}, \tag{A.5}$$

to get for the Schouten scalar  $J := \frac{R}{2(d-1)}$  and  $u := e^{(d/2-1)\sigma}$  the Yamabe equation

$$[\Delta + (d/2 - 1)J]u = (d/2 - 1) \widehat{J} u^{\frac{d+2}{d-2}}. \tag{A.6}$$

The trick (due to Branson) is now to slip in a 1 to rewrite as

$$\Delta(e^{(d/2-1)\sigma} - 1) + (d/2 - 1)J e^{(d/2-1)\sigma} = (d/2 - 1) \widehat{J} e^{(d/2+1)\sigma} \tag{A.7}$$

and take now the limit  $d \rightarrow 2$  that results in the PGC eq.

$$e^{2\sigma} \widehat{J} = J + \Delta\sigma. \tag{A.8}$$

The very same trick applied now to the higher-order Yamabe eq. based on the GJMS operators

$$P_{2k} u = \nabla \cdot S_k \nabla (u - 1) + (d/2 - k) Q_{2k}^d u = (d/2 - k) \widehat{Q}_{2k}^d u^{\frac{d+2k}{d-2k}} \tag{A.9}$$

with  $u = e^{(d/2-k)\sigma}$ , in the limit  $d \rightarrow 2k$  renders the higher (even-)dimensional generalization of the PGC eq.

$$e^{d\sigma} \widehat{Q} = Q + P\sigma, \tag{A.10}$$

where  $Q := Q_d^d$  and  $P := P_d$ .

Among the properties of the Q-curvature, the conformal invariance of its volume integral easily follows.

## B. Q-curvature and volume renormalization

Let the bulk metric to be that of an conformally compact (asymptotically) Einstein manifold, i.e.  $Ric(g_+) = -d g_+$ . The bulk geometry can be partially reconstructed by an asymptotic expansion, which is essentially the content of the Fefferman-Graham theorem [10]. One can always find local coordinates near the boundary (at  $r = 0$ ) to write the bulk metric as

$$g_+ = r^{-2} \{ dr^2 + g_r \}. \tag{B.1}$$

Euclidean  $AdS_{d+1}$  corresponds to the choice  $g_r = (1 - r^2)^2 g_0$  with  $4g_0$  being the round metric on the sphere  $S^d$ . The “reconstruction” theorem leads to the asymptotics

$$d \text{ odd :} \\ g_r = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + g^{(d)}r^d + \dots \tag{B.2}$$

$$d \text{ even :} \\ g_r = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + g^{(d)}r^d + hr^d \log r + \dots \tag{B.3}$$

where  $g^{(0)} = g$  is the chosen metric at the conformal boundary. For odd  $d$ ,  $g^{(j)}$  are tensors on the boundary and  $g^{(d)}$  is trace-free. For  $0 \leq j \leq d - 1$ ,  $g^{(j)}$  are locally formally determined by the conformal representative but  $g^{(d)}$  is formally undetermined, subject to the trace-free condition. For even  $d$ ,  $g^{(j)}$  are locally determined for  $j$  even  $0 \leq j \leq d - 2$ ,  $h$  is locally determined and trace-free. The trace of  $g^{(d)}$  is locally determined, but its trace-free part is formally undetermined. All this is dictated by Einstein equations.

The volume element has then an asymptotic expansion

$$dv_{g_+} = \sqrt{\frac{\det g_r}{\det g}} \frac{dv_g dr}{r^{d+1}} \\ = \{1 + v^{(2)}r^2 + (\text{even powers}) + v^{(d)}r^d + \dots\} \frac{dv_g dr}{r^{d+1}}, \tag{B.4}$$

where all coefficients  $v^{(j)}$ ,  $j = 1 \dots d$  are locally determined in term of curvature invariants of the boundary metric and  $v^{(d)} = 0$  if  $d$  is odd.

Before taking the  $r$ -integral, a regularization is needed. Then a subtraction (renormalization) prescription renders a finite answer when the regulator is removed. When  $d$  is odd, the finite remnant  $\mathcal{V}$  in the expansion (renormalized volume) turns out to be independent of the conformal choice of the boundary metric. If  $d$  is even, in turn,  $\mathcal{V}$  is no longer invariant and its variation under a Weyl transform of the boundary metric gives rise to the *conformal anomaly*. It is the coefficient  $\mathcal{L}$  of the  $\log \epsilon$ -term in the Hadamard (cutoff) regularization or residue at the pole in dimensional and Riesz regularizations

$$\mathcal{L} = \int_{\mathcal{M}} v^{(d)} dv_g, \tag{B.5}$$

given by the integral of a local curvature expression on the boundary, the invariant one in this case. The variation of  $\mathcal{V}$  happens to be connected to this invariant:  $g \rightarrow e^{2w}g$  for infinitesimal  $w$  makes  $\mathcal{V} \rightarrow \mathcal{V} + \int w v^{(d)} dv_g$  in the Hadamard regularization scheme.

The Q-curvature enters here and provides one of the important terms in volume renormalization asymptotics at conformal infinity [25]

$$\mathcal{L} = 2 c_{d/2} \int_{\mathcal{M}} Q_d dv_g. \tag{B.6}$$

Therefore, the Q-curvature is then proportional to the  $v^{(d)}$  coefficient in the volume expansion, up to total-derivative terms which are explicitly given by the “holographic formula” [26].

### C. Rigid case in dimensional regularization

The rigid computation in the bulk involves the volume renormalization of the ball model of the hyperbolic space with the standard metric:

$$-\log Z_{\text{grav}}^+/Z_{\text{grav}}^- = \left[ \int_0^\nu dx \, 2x \mathcal{A}_d(x) \right] \cdot \int_{\mathbb{H}_{d+1}} d\text{vol} . \quad (\text{C.1})$$

The factor in square brackets comes from the difference of the one-loop effective potentials associated to the two asymptotic behaviors, whose short distance divergences cancel out to render a finite result with

$$\mathcal{A}_d(\nu) = \frac{1}{2\nu} \frac{1}{2^d \pi^{\frac{d}{2}}} \frac{(\nu)^{\frac{d}{2}} (-\nu)^{\frac{d}{2}}}{\left(\frac{1}{2}\right)^{\frac{d}{2}}} \quad (\text{C.2})$$

conveniently written in terms of the Pochhammer symbol  $(a)_b := \Gamma(a+b)/\Gamma(a)$ .

The boundary computation on the standard sphere  $\mathbb{S}^d$ , expanding in spherical harmonics, results in an UV-divergent sum

$$-2 \log Z_\beta/Z_\alpha = \sum_{l=0}^{\infty} \text{deg}(d, l) \log \frac{\Gamma(l + \frac{d}{2} + \nu)}{\Gamma(l + \frac{d}{2} - \nu)} . \quad (\text{C.3})$$

Here we have a weighted sum with the degeneracies of the spherical harmonics

$$\text{deg}(d, l) = \frac{2l + d - 1}{d - 1} \frac{(d - 1)_l}{l!} , \quad (\text{C.4})$$

and the ratio  $\frac{\Gamma(l + \frac{d}{2} + \nu)}{\Gamma(l + \frac{d}{2} - \nu)}$  are nothing but the eigenvalues of the *intertwiner* (cf. eq.2.13 in [13]) between conjugate representations (with conformal labels  $\lambda_-$  and  $\lambda_+$ ), which is the two-point function  $\langle O_\beta O_\beta \rangle$  on the round sphere (see e.g. [75]).

We extended the mapping from that of the integrated anomaly to the renormalized partition functions as well. The anomaly can be read as the residue of the pole term

$$\mathcal{L}_{d+1} \cdot \mathcal{A}_d(\nu) , \quad (\text{C.5})$$

and the renormalized determinant is given by

$$-\log Z_{\text{grav}}^+/Z_{\text{grav}}^- = \left[ \int_0^\nu dx \, 2x \mathcal{A}_d(x) \right] \cdot \mathcal{V}_{d+1} + \left[ \int_0^\nu dx \, 2x \mathcal{B}_d(x) \right] \cdot \mathcal{L}_{d+1} , \quad (\text{C.6})$$

where

$$\mathcal{B}_d(\nu) = \frac{\mathcal{A}_d(\nu)}{2} \left\{ \log(4\pi) + \psi \left( \frac{1}{2} - \frac{d}{2} \right) - \psi \left( \frac{d}{2} + \nu \right) - \psi \left( \frac{d}{2} - \nu \right) \right\} . \quad (\text{C.7})$$

## References

- [1] E. Cunningham, *The principle of relativity in electrodynamics and an extension thereof*, *Proc. London Math. Soc.* **8** (1910) 77.
- [2] H. Bateman, *The transformations of the electrodynamical equations*, *Proc. London Math. Soc.* **8** (1910) 223.
- [3] W. Pauli, *Über die Invarianz der Dirac'schen Wellengleichungen gegenüber Ähnlichkeitstransformationen des Linienelementes im Fall verschwindender Ruhemasse*, *Helv. Phys. Acta* **13** (1940) 204.
- [4] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, *Osaka J. Math.* **12** (1960) 21.
- [5] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, MIT preprint, (1983).
- [6] M.G. Eastwood and M. Singer, *A conformally invariant Maxwell gauge*, *Phys. Lett. A* **107** (1985) 73.
- [7] R.J. Riegert, *A nonlocal action for the trace anomaly*, *Phys. Lett. B* **134** (1984) 56.
- [8] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett. B* **103** (1981) 207.
- [9] C.R. Graham, R. Jenne, L.J. Manson and G.A.J. Sparling, *Conformally invariant powers of the Laplacian. I. Existence*, *J. London Math. Soc.* **46** (1992) 557.
- [10] C. Fefferman and C.R. Graham, *Conformal invariants*, in *Elie Cartan et les Mathématiques d'Aujourd'hui*, Astérisque (1985) 95.
- [11] R. Manvelyan and D.H. Tehrakian, *Conformal coupling of the scalar field with gravity in higher dimensions and invariant powers of the Laplacian*, *Phys. Lett. B* **644** (2007) 370 [[hep-th/0611077](#)].
- [12] R. Manvelyan, K. Mkrtchyan and R. Mkrtchyan, *Conformal invariant powers of the Laplacian, Fefferman-Graham ambient metric and Ricci gauging*, [arXiv:0707.1737](#).
- [13] T. Branson, *The functional determinant*, Global Analysis Research Center Lecture Note Series, Number 4, Seoul National University, Seoul South Korea (1993); *Sharp inequalities, the functional determinant, and the complementary series*, *Trans. Am. Math. Soc.* **347** (1995) 3671.
- [14] S. Deser and A. Schwimmer, *Geometric classification of conformal anomalies in arbitrary dimensions*, *Phys. Lett. B* **309** (1993) 279 [[hep-th/9302047](#)].
- [15] S. Alexakis, *On the decomposition of global conformal invariants I*, [math.DG/0509571](#); *On the decomposition of global conformal invariants II*, *Adv. Math.* **206** (2006) 466 [[math.DG/0509572](#)]; *The decomposition of global conformal invariants: on a conjecture of Deser and Schwimmer*, [arXiv:0711.1685](#).
- [16] N. Boulanger, *General solutions of the Wess-Zumino consistency condition for the Weyl anomalies*, *JHEP* **07** (2007) 069 [[hep-th/07042472](#)]; *Algebraic classification of Weyl anomalies in arbitrary dimensions*, *Phys. Rev. Lett.* **98** (2007) 261302 [[hep-th/07060340](#)].
- [17] T. Branson, P. Gilkey and J. Pohjanpelto, *Invariants of locally conformally flat manifolds*, *Trans. Am. Math. Soc.* **347** (1995) 939.



- [18] C.R. Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, [math.DG/9909042](#).
- [19] J.M. Maldacena, *The large- $N$  limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [20] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [[hep-th/9802109](#)].
- [21] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [22] C.R. Graham and J. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, *Adv. Math.* **87** (1991) 186.
- [23] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, [hep-th/9805114](#).
- [24] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)]; *Holography and the Weyl anomaly*, *Fortschr. Phys.* **48** (2000) 125 [[hep-th/9812032](#)].
- [25] C.R. Graham and M. Zworski, *Scattering matrix in conformal geometry*, *Invent. Math.* **152** (2003) 89 [[math.DG/0109089](#)].
- [26] C.R. Graham and A. Juhl, *Holographic formula for  $Q$ -curvature*, *Adv. Math.* **216** (2007) 841 [[arXiv:0704.1673](#)].
- [27] C. Fefferman and C.R. Graham,  *$Q$ -curvature and Poincaré metrics*, *Math. Res. Lett.* **9** (2002) 139 [[math.DG/0110271](#)].
- [28] A. Petkou and K. Skenderis, *A non-renormalization theorem for conformal anomalies*, *Nucl. Phys.* **B 561** (1999) 100 [[hep-th/9906030](#)].
- [29] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [[hep-th/0002230](#)].
- [30] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, [hep-th/0404176](#).
- [31] F. Bastianelli, S. Frolov and A.A. Tseytlin, *Conformal anomaly of  $(2,0)$  tensor multiplet in six dimensions and AdS/CFT correspondence*, *JHEP* **02** (2000) 013 [[hep-th/0001041](#)].
- [32] S. Nojiri, S.D. Odintsov and S. Ogushi, *Holographic renormalization group and conformal anomaly for AdS<sub>9</sub>/CFT<sub>8</sub> correspondence*, *Phys. Lett.* **B 500** (2001) 199 [[hep-th/0011182](#)].
- [33] S.S. Gubser and I. Mitra, *Double-trace operators and one-loop vacuum energy in AdS/CFT*, *Phys. Rev.* **D 67** (2003) 064018 [[hep-th/0210093](#)].
- [34] S.S. Gubser and I.R. Klebanov, *A universal result on central charges in the presence of double-trace deformations*, *Nucl. Phys.* **B 656** (2003) 23 [[hep-th/0212138](#)].
- [35] S.S. Gubser, *Loop effects in AdS/CFT and beyond*, *Prog. Theor. Phys. Suppl.* **152** (2004) 30.
- [36] T. Hartman and L. Rastelli, *Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT*, *JHEP* **01** (2008) 019 [[hep-th/0602106](#)].
- [37] D.E. Diaz and H. Dorn, *Partition functions and double-trace deformations in AdS/CFT*, *JHEP* **05** (2007) 046 [[hep-th/0702163](#)].

- [38] M.T. Anderson, *Geometric aspects of the AdS/CFT correspondence*, in *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, O. Biquard ed., *Euro. Math. Soc. Zurich* (2005) 1 [[hep-th/0403087](#)].
- [39] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys. (NY)* **144** (1982) 249.
- [40] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [[hep-th/9905104](#)].
- [41] E. Witten, *Multi-trace operators, boundary conditions and AdS/CFT correspondence*, [hep-th/0112258](#);  
M. Berkooz, A. Sever and A. Shomer, *Double-trace deformations, boundary conditions and spacetime singularities*, *JHEP* **05** (2002) 034 [[hep-th/0112264](#)];  
W. Mueck, *An improved correspondence formula for AdS/CFT with multi-trace operators*, *Phys. Lett.* **B 531** (2002) 301 [[hep-th/0201100](#)];  
P. Minces, *Multi-trace operators and the generalized AdS/CFT prescription*, *Phys. Rev.* **D 68** (2003) 024027 [[hep-th/0201172](#)].
- [42] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical  $O(N)$  vector model*, *Phys. Lett.* **B 550** (2002) 213 [[hep-th/0210114](#)].
- [43] D.E. Diaz and H. Dorn, *On the AdS higher spin/ $O(N)$  vector model correspondence: degeneracy of the holographic image*, *JHEP* **07** (2006) 022 [[hep-th/0603084](#)].
- [44] R. Camporesi, *Zeta function regularization of one loop effective potentials in anti-de Sitter space-time*, *Phys. Rev.* **D 43** (1991) 3958.
- [45] A.R. Gover, *Laplacian operators and Q-curvature on conformally Einstein manifolds*, [math.DG/0506037](#).
- [46] T.P. Branson and B. Ørsted, *Conformal indices of Riemannian manifolds*, *Compos. Math.* **60** (1986) 261.
- [47] W.I. Weisberger, *Normalization of the path integral measure and the coupling constants for bosonic strings*, *Nucl. Phys.* **B 284** (1987) 171.
- [48] J.S. Dowker and K. Kirsten, *The Barnes zeta-function, sphere determinants and Glaisher-Kinkelin-Bendersky constants*, *Anal. Appl.* **3** (2005) 45 [[hep-th/0301143](#)].
- [49] L. Bendersky, *Sur la fonction gamma généralisée*, *Acta Math.* **61** (1933) 263.
- [50] J. Choi, H.M. Srivastava and V.S. Adamchik, *Multiple gamma and related functions*, *Appl. Math. and Comp.* **134** (2003) 515.
- [51] C. Guillarmou, *Generalized Krein formula, determinants and Selberg zeta function in even dimension*, [math.SP/0512173](#).
- [52] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, *Diffeomorphisms and holographic anomalies*, *Class. and Quant. Grav.* **17** (2000) 1129 [[hep-th/9910267](#)].
- [53] A. Schwimmer and S. Theisen, *Entanglement entropy, trace anomalies and holography*, *Nucl. Phys.* **B 801** (2008) 1 [[arXiv:0802.1017](#)].
- [54] A. Chang, J. Qing and P. Yang, *On the renormalized volumes for conformally compact Einstein manifolds*, [math.DG/0512376](#).

- [55] R. Olea, *Regularization of odd-dimensional AdS gravity: kounterterms*, *JHEP* **04** (2007) 073 [[hep-th/0610230](#)].
- [56] S. Carlip, *Dynamics of asymptotic diffeomorphisms in  $(2 + 1)$ -dimensional gravity*, *Class. and Quant. Grav.* **22** (2005) 3055 [[gr-qc/0501033](#)].
- [57] R. Aros, M. Romo and N. Zamorano, *Conformal gravity from AdS/CFT mechanism*, *Phys. Rev.* **D 75** (2007) 067501 [[hep-th/0612028](#)].
- [58] T. Branson and B. Ørsted, *Explicit functional determinants in four dimensions*, *Proc. Am. Math. Soc.* **113** (1991) 669.
- [59] P.B. Gilkey, *Recursion relations and the asymptotic behavior of the eigenvalues of the Laplacian*, *Compos. Math.* **38** (1979) 201.
- [60] T. Branson, *An anomaly associated with 4-dimensional quantum gravity*, *Commun. Math. Phys.* **178** (1996) 301.
- [61] P.B. Gilkey, *The spectral geometry of the higher order Laplacian*, *Duke Math. J.* **47** (1980) 511 [*Erratum ibid.* **48** (1980) 887].
- [62] T. Branson and L.J. Peterson, unpublished.
- [63] I.G. Avramidi, *Covariant methods for the calculation of the effective action in quantum field theory and investigation of higher-derivative quantum gravity*, [hep-th/9510140](#).
- [64] A. Cappelli and G. D’Appollonio, *On the trace anomaly as a measure of degrees of freedom*, *Phys. Lett.* **B 487** (2000) 87 [[hep-th/0005115](#)].
- [65] S. Paycha and S. Rosenberg, *Conformal anomalies via canonical traces*, [math.DG/0508290](#).
- [66] D. Anselmi, *Anomalies, unitarity and quantum irreversibility*, *Ann. Phys. (NY)* **276** (1999) 361 [[hep-th/9903059](#)].
- [67] D. Anselmi, *Quantum irreversibility in arbitrary dimension*, *Nucl. Phys.* **B 567** (2000) 331 [[hep-th/9905005](#)].
- [68] A. Juhl, private communication (2008).
- [69] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577.
- [70] S. Nojiri and S.D. Odintsov, *The one-loop vacuum energy and RG flow induced by double-trace operators in AdS/CFT and dS/CFT correspondence*, *Phys. Rev.* **D 69** (2004) 023511 [[hep-th/0302054](#)].
- [71] S.J. Patterson and P.A. Perry, *The divisor of the Selberg zeta function for Kleinian groups*, *Duke Math. J.* **106** (2001) 321.
- [72] P.A. Perry and F. Williams, *Selberg zeta function and trace formula for the BTZ black hole*, *Int. J. Pure Appl. Math.* **9** (2003) 1.
- [73] G.W. Moore, *Les Houches lectures on strings and arithmetic*, [hep-th/0401049](#).
- [74] I.M. Gelfand and G.E. Shilov, *Generalized functions*, vol. 1, Academic Press, New York U.S.A. (1964).
- [75] V.K. Dobrev, *Intertwining operator realization of the AdS/CFT correspondence*, *Nucl. Phys.* **B 553** (1999) 559 [[hep-th/9812194](#)].